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Electronic Notes in Theoretical Computer Science 50 No. 2 (2001) – Proc. BOTH 2001
URL: http://www.elsevier.nl/locate/entcs/volume50.html 14 pages

Towards Lambda Calculus Order-Incompleteness

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Abstract

After Scott, mathematical models of the type-free lambda calculus are constructed by order theoretic methods and classified into semantics according to the nature of their representable functions. Selinger [48] asked if there is a lambda theory that is not induced by any non-trivially partially ordered model (order-incompleteness problem). In terms of Alexandroff topology (the strongest topology whose specialization order is the order of the considered model) the problem of order-incompleteness can be also characterized as follows: a lambda theory T is order-incomplete if, and only if, every partially ordered model of T is partitioned by the Alexandroff topology in an infinite number of connected components (= minimal upper and lower sets), each one containing exactly one element of the model. Towards an answer to the order-incompleteness problem, we give a topological proof of the following result: there exists a lambda theory whose partially ordered models are partitioned by the Alexandroff topology in an infinite number of connected components, each one containing at most one λ-term denotation. This result implies the incompleteness of every semantics of lambda calculus given in terms of partially ordered models whose Alexandroff topology has a finite number of connected components (e.g. the Alexandroff topology of the models of the continuous, stable and strongly stable semantics is connected).

1 Introduction

Many familiar models of the type-free lambda calculus are constructed by order theoretic methods. Computational motivations and intuitions justified Scott's view of models (see [42] [43]) as partially ordered sets (sets of observations or informations) and of functions as monotonic functions over these sets. After Scott, a large number of mathematical models for the lambda calculus, arising from syntax-free constructions, have been introduced in various

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categories of domains (see [1] [46]) and classified into semantics according to the nature of their representable functions (see [2] [3] [4] [9] [15] [19] [24]). Scott's continuous semantics [43] is given in the category whose objects are complete partial orders and morphisms are continuous functions. The stable semantics introduced by Berry in [10] and the strongly stable semantics introduced by Bucciarelli and Ehrhard in [11] are strengthening of the continuous semantics. The stable semantics is given in the category of DI-domains with stable functions as morphisms, while the strongly stable one in the category of DI-domains with coherence, and strongly stable functions as morphisms.

Lambda theories are consistent extensions of the lambda calculus that include β-conversion. They arise by syntactical considerations, a lambda theory may correspond to a possible operational (observational) semantics of lambda calculus (see e.g. [2] [3] [23]), as well as by semantic ones, a lambda theory may be the theory of a model of lambda calculus (see e.g. [3] [9]). The problem of the completeness/incompleteness of a semantics can be stated as follows: are the set of the lambda theories determined by a semantics equal or strictly included within the set of consistent lambda theories?

The first incompleteness result was obtained by Honsell and Ronchi della Rocca [24] for the continuous semantics via a hard syntactical proof. Gouy [20] proved the incompleteness of the stable semantics with a much harder syntactical proof. Other more semantic proofs of incompleteness for the continuous and stable semantics can be found in [7]. Bastonero [6] provides an incompleteness result for the hypercoherence semantics.

Recently, the author has introduced in [37] a new technique to prove the incompleteness of a wide range of lambda calculus semantics (including the strongly stable one, whose incompleteness had been conjectured). Roughly, the technique used in [37] for proving that a class $\mathcal{C}$ of models is incomplete is the following. We remark that the partially ordered models of the lambda calculus are topological combinatory algebras w.r.t. the Alexandroff topology (the strongest topology whose specialization order is the order of the considered model). Then we find a (topological) property P verified by all models in $\mathcal{C}$ and find a lambda theory whose models do not verify P. The technique was applied to the models of lambda calculus based on domains (continuous, stable, strongly stable models in particular). These models satisfy a strong property of connectedness, while we found a lambda theory whose models satisfy an orthogonal property of separation.

The problem of the incompleteness of the semantics of lambda calculus is also related to the open problem of the order-incompleteness of the lambda theories. Selinger [48] asked if there is a lambda theory that is not induced by any non-trivially partially ordered model. He gave a syntactical characterization, in terms of so-called generalized Mal'cev operators, of the order-incomplete lambda theories. Roughly, the problem of the order-incompleteness can be stated as follows: does it exist a sequence $M_1, \dots, M_n$

of closed λ-terms such that the lambda theory $\mathcal{T}_n$, axiomatized by

$$x = M_1 xyy;\ M_i xxy = M_{i+1} xyy;\ M_n xxy = y\ (1 \leq i < n),$$

is consistent? Plotkin and Simpson (see [47]) have shown that $\mathcal{T}_1$ is inconsistent, while Plotkin and Selinger (see [47]) obtained the same result for $\mathcal{T}_2$. It is an open problem whether $\mathcal{T}_n$ $(n \geq 3)$ can be consistent. Order-incompleteness is also related to Plotkin's conjecture (see [36] [47] [48]) about the existence of absolutely unorderable combinatory algebras, where a combinatory algebra is absolutely unorderable if it cannot be embedded in any orderable combinatory algebra.

The problem of order-incompleteness can be also characterized in terms of Alexandroff topology. A lambda theory $\mathcal{T}$ is order-incomplete if, and only if, the Alexandroff topology of any partially ordered model of $\mathcal{T}$ is the discrete topology if, and only if, the Alexandroff topology of any partially ordered model of $\mathcal{T}$ partitions the model in an infinite number of connected components (= minimal upper and lower sets), each one containing exactly one element of the model. Towards an answer to the order-incompleteness problem, in this paper we give a topological proof of the following result: there exists a lambda theory whose partially ordered models are partitioned by the Alexandroff topology in an infinite number of connected components, each one containing at most one λ-term denotation. This result implies the incompleteness, that had been conjectured in [37], of every semantics of lambda calculus given in terms of partially ordered models whose Alexandroff topology has a finite number of connected components (e.g. the Alexandroff topology of continuous, stable and strongly stable semantics is connected).

2 Preliminaries

To keep this article self-contained, we summarize some definitions and results that we will need in the subsequent part of the paper. With regard to the lambda calculus we follow the notation and terminology of Barendregt (see [3]).

For the general theory of lambda calculus the reader may consult Barendregt [3] and Krivine [28]. For the general theory of universal algebras the reader may consult Burris and Sankappanavar [12] Gratzer [21] and McKenzie, McNulty and Taylor [29]. The main references for topological algebras are Taylor [50] [51], Gumm [22], Bentz [8] and Coleman [13] [14].

2.1 Lambda theories

Λ denotes the set of λ-terms, while Λ^o denotes the set of closed λ-terms, where a λ-term is closed if it does not admit free occurrences of variables.

Lambda theories are consistent extensions of the lambda calculus that are closed under derivation. Remember that an equation is a formula of the form

$M = N$ with $M, N \in \Lambda$. The equation is closed if M and N are closed λ-terms. If $\mathcal{T}$ is a set of equations, then the theory $\lambda + \mathcal{T}$ is obtained by adding to the axioms and rules of the lambda calculus the equations in $\mathcal{T}$ as new axioms. If $\mathcal{T}$ is a set of closed equations, $\mathcal{T}^+$ is the set of closed equations provable in $\lambda + \mathcal{T}$. $\mathcal{T}$ is a lambda theory if $\mathcal{T}^+ = \mathcal{T}$ (see [3, Def. 4.1.1]). As a matter of notation, $\mathcal{T} \vdash M = N$ stands for $\lambda + \mathcal{T} \vdash M = N$; this is also written as $M =_{\mathcal{T}} N$. $[M]^o_{\mathcal{T}} = \{N \in \Lambda^o : \mathcal{T} \vdash N = M\}$ denotes the equivalence class of the closed λ-term M.

2.2 Combinatory algebras and λ-models

An algebra $\mathbf{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$, where $\cdot$ is a binary operation and $\mathbf{k}, \mathbf{s}$ are constants, is called a *combinatory algebra* (Curry [16], Schönfinkel [41]) if it satisfies the following identities (as usual the symbol $\cdot$ is omitted, and association is to the left): $\mathbf{k}xy = x; \mathbf{s}xyz = xz(yz)$. In the equational language of combinatory algebras the derived combinator $\mathbf{1}$ is defined as $\mathbf{1} \equiv \mathbf{s}(\mathbf{ki})$. A function $f : C \to C$ is called *representable* if there exists an element $c \in C$ such that $cz = f(z)$ for all $z \in C$. If this last condition is satisfied, we say that c represents map f in $\mathbf{C}$.

Let $\mathbf{C}$ be a combinatory algebra and let $\bar{c}$ be a new symbol for each $c \in C$. Extend the language of lambda calculus by adjoining $\bar{c}$ as a new constant symbol for each $c \in C$. Let $\Lambda^o(C)$ be the set of closed λ-terms with constants from C. The interpretation of terms in $\Lambda^o(C)$ with elements of C can be defined by induction as follows (for all $M, N \in \Lambda^o(C)$ and $c \in C$):

$$|\bar{c}|_{\mathbf{C}} = c;\ \ |(MN)|_{\mathbf{C}} = |M|_{\mathbf{C}}|N|_{\mathbf{C}};\ \ |\lambda x.M|_{\mathbf{C}} = \mathbf{1}m,$$

where $m \in C$ is any element representing the following map $f : C \to C$:

$$f(c) = |M[x := \bar{c}]|_{\mathbf{C}}, \quad \text{for all } c \in C.$$

The drawback of the previous definition is that, if $\mathbf{C}$ is an arbitrary combinatory algebra, it may happen that map f is not representable. The axioms of a subclass of combinatory algebras, called *λ-models* or models of lambda calculus (Meyer [30], Scott [45], [3, Def. 5.2.7]), were expressly chosen to make coherent the previous definition of interpretation. For every λ-model $\mathbf{C}$, the set $Th(\mathbf{C}) = \{M = N : M, N \in \Lambda^o,\ \mathbf{C} \models M = N\}$ constitutes a lambda theory. $\mathbf{C}$ is a model of the lambda theory $\mathcal{T}$ if $\mathcal{T} = Th(\mathbf{C})$.

We would like to point out here that there exists an algebraic approach to the model theory of lambda calculus, alternative to combinatory logic, that allows to keep the lambda notation and all the functional intuitions (see [31] [32] [33] [38] [39] [40]).

2.3 Topology

If (A, τ) is a topological space (we will occasionally avoid explicit mention of τ) then the closure of a subset U of A will be denoted by $\overline{U}$ (if $U = \{b\}$ is a singleton set, then we write $\overline{b}$ for $\overline{\{b\}}$). Recall that $a \in \overline{U}$ if $U \cap V \neq \emptyset$ for every open neighborhood V of a.

For any space (A, τ) a preorder can be defined by

$$a \leq_\tau b \text{ iff } a \in \overline{b} \text{ iff } \forall U \in \tau (a \in U \Rightarrow b \in U).$$

We have

$$\tau \text{ is } T_0 \text{ iff } \leq_\tau \text{ is a partial order.}$$

For any T_0-space A the partial order $\leq_\tau$ is called *the specialization order* of τ. Notice that any continuous map between T_0-spaces is necessarily monotone and that the order is discrete (i.e. satisfies $a \leq_\tau b$ iff $a = b$) iff A is a T_1-space.

A space A is T_2 (or Hausdorff) if for all $a, b \in A$ there exist open sets U and V with $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

The previous axioms of separation can be relativized to pairs of elements. For example, a and b are T_2-separable, if there exist open sets U and V with $a \in U$, $b \in V$ and $U \cap V = \emptyset$. T_1-, T_0-separability are similarly defined.

The connected component of an element a of a space A is the greatest connected subset of A including a. The connected components define a partition of the space A.

Each partition P of any set X into disjoint subsets, together with $\emptyset$, is a basis for a topology on X, known as a *partition topology.* A subset of X is then open if and only if it is the union of sets belonging to P and thus its complement is also open; thus a set is open iff it is closed. The trivial partitions yield the discrete or indiscrete topologies. In any other cases X with a partition topology is not T_0.

Let $(A, \leq)$ be a partially ordered set (poset). $B \subseteq A$ is an upper (lower) set if $b \in B$ and $b \leq a$ ($a \leq b$) imply $a \in B$. We utilize the notations $B\uparrow$ ($B\downarrow$, $B\updownarrow$ respectively) for the least upper (lower, upper and lower) set containing a subset B of A. We write $a\uparrow$ ($a\downarrow$, $a\updownarrow$ respectively) for $\{a\}\uparrow$ ($\{a\}\downarrow$, $\{a\}\updownarrow$).

Given a poset $(A, \leq)$ we can find many T_0-topologies τ on A for which $\leq$ is the specialization ordering of τ (see Johnstone [25, Section II.1.8]). The Alexandroff topology and the weak topology defined below are the maximal one and the minimal one with this property.

The *Alexandroff topology* $a_\leq$ is constituted by the collection of all upper sets in A, i.e.,

$$U \text{ is an Alexandroff open (A-open, for short) iff } U = U\uparrow.$$

Then $a\uparrow$ is the least open set containing a. A subset U is an Alexandroff closed set (A-closed set, for short) iff $U = U\downarrow$. A function is continuous w.r.t.

the Alexandroff topology if, and only if, it is monotone. Every Alexandroff space is T_0.

The *weak topology* $w_\leq$ is constituted by the smallest topology for which all sets of the form $a{\downarrow}$ are closed, i.e. the topology based by sets of the form $A - (a_1{\downarrow} \cup \ldots \cup a_k{\downarrow})$.

Let $(A, \leq)$ be a poset, τ be a topology on A. Then τ is T_0 with specialization order $\leq$ if, and only if, $w_\leq \subseteq \tau \subseteq a_\leq$.

3 The topological theorem

Separation axioms in topology stipulate the degree to which distinct points may be separated by open sets or by closed neighborhoods of open sets. In the main theorem of this Section we prove that every partially ordered combinatory algebra, under very weak hypotheses, admits elements which can be separated in a very strong way.

Let $(A, \leq)$ be a poset with Alexandroff topology $a_\leq$. The intersection of every family of A-open sets is A-open; thus the union of every family of A-closed sets is A-closed. This is a consequence of fact that, for every subset V of a poset $(A, \leq)$, there exist a least upper set $V{\uparrow}$ and a least lower set $V{\downarrow}$, all of them including V. It follows that the family of A-closed sets of the Alexandroff topology associated with a poset $(A, \leq)$ is the Alexandroff topology $a_\geq$ associated with the poset $(A, \geq)$.

We now consider the clopen sets, i.e., the sets which are contemporaneously A-open and A-closed:

$$X \text{ is A-clopen iff } X = X{\updownarrow} = X{\uparrow} = X{\downarrow}.$$

Notice that a connected component is a closed set in every topological space. For the Alexandroff topology we have that a subset of a poset is a connected component w.r.t. $a_\leq$ iff it is such w.r.t. $a_\geq$. So a connected component is a minimal A-clopen set, and the minimal A-clopen sets constitute the partition of the Alexandroff space in connected components. In terms of partial ordering they can be described as follows. Let $\sim_\leq$ be the symmetric closure of $\leq$, i.e.,

$$a \sim_\leq b \text{ iff either } a \leq b \text{ or } b \leq a.$$

The equivalence relation $\approx_\leq$ on A generated by $\leq$ is defined as follows:

$$a \approx_\leq b \;\Leftrightarrow\; (\exists c_0, \ldots, c_n)\; a = c_0 \sim_\leq c_1 \sim_\leq \ldots \sim_\leq c_{n-1} \sim_\leq c_n = b.$$

Then the equivalence classes of $\approx_\leq$ are the partition of A in connected components w.r.t. the Alexandroff topology, i.e.,

$$\approx_\leq\text{-equivalence class} = \text{minimal A-clopen set} = \text{connected component}.$$

It is an easy matter to verify that the A-clopen sets of an Alexandroff space constitute a topology, denoted by $\pi_{\leq}$. It is the partition topology (see Section 2.3) generated by the partition of the space in connected components.

Since a map is monotone iff the inverse image of an upper set is an upper set iff the inverse image of a lower set is a lower set, then every monotone map is continuous w.r.t. the partition topology $\pi_{\leq}$.

A *partially ordered combinatory algebra*, a po-combinatory algebra for short, is a pair $(\mathbf{C}, \leq)$ where $\mathbf{C}$ is a combinatory algebra and $\leq$ is a partial order on C which makes the application operator of $\mathbf{C}$ monotone.

An Alexandroff combinatory algebra is a pair $(\mathbf{A}, a_{\leq})$ where $\mathbf{A}$ is a combinatory algebra and $a_{\leq}$ is an Alexandroff topology on the underlying set A with the property that the application operator of $\mathbf{A}$ is continuous (= monotone) with respect to $a_{\leq}$.

The category of po-combinatory algebras with monotone maps as morphisms and the category of Alexandroff combinatory algebras with continuous maps as morphisms are equivalent.

In the following we always assume defined on a po-combinatory algebra the Alexandroff topology.

In the following theorem we prove that every po-combinatory algebra under very weak hypotheses admits elements which can be separated by A-clopen sets.

Theorem 3.1 *Let $(\mathbf{A}, \leq)$ be a po-combinatory algebra for which there exist a combinatory term $s(x, y)$ and a constant 0 such that*

$$s(x, x) = 0.$$

For all $a, b \in A$, define a sequence of elements of A as follows:

$$c_1 = s(a, b); \quad c_{n+1} = s(c_n, 0).$$

If $c_n \neq 0$ for all n, then there exist an A-clopen set V such that $a \in V$ and $b \notin V$.

Proof. The proof is divided in claims.

Claim 3.2 *If c_1 and 0 are T_2-separated w.r.t. the partition topology $\pi_{\leq}$ then a and b are also T_2-separated w.r.t. the partition topology $\pi_{\leq}$.*

Let U and S be two A-clopen sets such that U is a neighbourhood of c_1, S is a neighbourhood of 0, and $U \cap S = \emptyset$. Because $s(a, b) = c_1 \in U$ and s is continuous w.r.t. $\pi_{\leq}$, then there exist two A-clopen sets V and W such that V is a neighbourhood of a, W is a a neighbourhood of b, and $s(V, W) \subseteq U$. If $d \in V \cap W$ then $0 = s(d, d) \in U$ contradicting the choice of U.

Claim 3.3 *For every element $z \in A$ define by induction the following sets:*

$$z^0 = \{z\}; \quad z^{2i+2} = (z^{2i+1})\!\downarrow; \quad z^{2i+1} = (z^{2i})\!\uparrow.$$

Then set $\cup_{i\geq 0} z^i$ *is equal to the least A-clopen set* $z\updownarrow$ *(= connected component) including* z*.*

It is sufficient to check that $\cup_{i\geq 0} z^i$ is an upper and lower set contained within $z\updownarrow$.

Claim 3.4 *For every* $k \geq 1$ *we have that*

$$s(c_k\uparrow, 0\uparrow) \subseteq c_{k+1}\uparrow.$$

The relation follows from the monotonicity of s and from the equality $c_{k+1} = s(c_k, 0)$.

Claim 3.5 *For every* $k \geq 1$ *we have that*

$$c_k \not\leq 0.$$

Assume, by the way of contradiction, that $c_k \leq 0$. Then by monotonicity we have that

$$0 = s(c_k, c_k) \leq s(c_k, 0) = c_{k+1}$$

and

$$c_{k+1} = s(c_k, 0) \leq s(0, 0) = 0.$$

This contradicts the hypothesis that $\leq$ is a partial order.

Claim 3.6 *For every* $k \geq 1$ *we have that* c_k *and* 0 *are incompatible, i.e.,*

$$c_k\uparrow \cap\, 0\uparrow = \emptyset.$$

If there exists an element b such that $b \geq c_k$ and $b \geq 0$ then by monotonicity we have that

$$c_{k+1} = s(c_k, 0) \leq s(b, b) = 0$$

that contradicts Claim 3.5.

Claim 3.7 *For every* $k \geq 1$ *and every* $i \geq 1$ *we have that*

$$c_k^i \cap 0^i = \emptyset; \quad s(c_k^i, 0^i) \subseteq c_{k+1}^i.$$

(see the definition of '$(-)^i$*' in Claim 3.3).*

For $i = 1$ the conclusion follows from Claim 3.6 and Claim 3.4. Assume the conclusion true for i and prove it for $i+1$. Let $s(c_k^i, 0^i) \subseteq c_{k+1}^i \subseteq c_{k+1}^{i+1}$. If i is odd c_{k+1}^i is A-open and c_{k+1}^{i+1} is A-closed. Since s is continuous the pre-image of the A-closed set c_{k+1}^{i+1} under the map s is A-closed. From $s(c_k^i, 0^i) \subseteq c_{k+1}^{i+1}$ the pre-image of c_{k+1}^{i+1}, that is closed, contains $c_k^i \times 0^i$, so $s(c_k^{i+1}, 0^{i+1}) \subseteq c_{k+1}^{i+1}$. If i is even we make the same reasoning by using the Alexandroff topology $a_{\geq}$ associated with the partial ordering $\geq$ on A.

We now prove that $c_k^{i+1} \cap 0^{i+1} = \emptyset$. Assume i odd so that c_k^{i+1} and 0^{i+1} are A-closed sets. Assume, by the way of contradiction, that there is $f \in c_k^{i+1} \cap 0^{i+1}$. It follows that $0 = s(f, f) \in c_{k+1}^{i+1}$, because we have already shown that $s(c_k^{i+1}, 0^{i+1}) \subseteq c_{k+1}^{i+1}$. But by definition of closure of a set this is possible only if for every A-open neighbourhood Z of 0, we have that $Z \cap c_{k+1}^i \neq \emptyset$. But this contradicts the induction hypothesis $c_{k+1}^i \cap 0^i = \emptyset$ because 0^i is an A-open neighbourhood of 0. A similar reasoning works for an even i by using the Alexandroff topology $a_\geq$ associated with the partial ordering $\geq$ on A.

Claim 3.8 *For every $k \geq 1$ we have that c_k and 0 are T_2-separated w.r.t. the partition topology $\pi_\leq$.*

The least clopen sets including c_k and 0 are respectively $\cup_{i\geq 0} c_k^i$ and $\cup_{i\geq 0} 0^i$. Then the conclusion follows from Claim 3.7.

Since c_1 and 0 are T_2-separated w.r.t. the partition topology $\pi_\leq$ from Claim 3.8, then the conclusion of the theorem follows from Claim 3.2. □

4 Incompleteness

In this Section we prove the main theorem of the paper.

Consider the (consistent and) semisensible lambda theory Π axiomatized by

$$\Omega xx = \Omega,$$

where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

Lemma 4.1

$$\Pi \vdash \Omega tu = \Omega \Leftrightarrow \Pi \vdash t = u.$$

Proof. Let $\to_\Pi$ be the following reduction rule:

$$(1) \qquad \Omega MN \to_\Pi \Omega$$

for every M and N such that $\Pi \vdash M = N$. The reflexive closure of $\to_\Pi$ satisfies the diamond property, and the relations $\twoheadrightarrow_\beta$ and $\twoheadrightarrow_\Pi$ commute. Then the reduction rule $\to_{\beta\Pi} = \to_\beta \cup \to_\Pi$ is Church-Rosser by the Hindley-Rosen Lemma (see Barendregt [3, Prop. 3.3.5]).

Then we prove that Π is the lambda theory generated by conversion $\cong_{\beta\Pi}$ from $\to_{\beta\Pi}$, i.e.,

$$(2) \qquad \Pi \vdash M = N \text{ iff } M \cong_{\beta\Pi} N.$$

Since $\Omega MN \to_\Pi \Omega$ iff $\Pi \vdash M = N$, then it is obvious that $M \cong_{\beta\Pi} N$ implies $\Pi \vdash M = N$. For the opposite direction, it is sufficient to consider that $\Omega xx \to_\Pi \Omega$ for the unique axiom $\Omega xx = \Omega$ of Π.

If $\Pi \vdash \Omega tu = \Omega$ then $\Omega tu \cong_{\beta\Pi} \Omega$, so that there is a reduction $\Omega tu \twoheadrightarrow_{\beta\Pi} \Omega$. This is possible only if Ωtu is a Π-redex i.e. if $\Pi \vdash t = u$. □

Lemma 4.2 *Let t and u be two λ-terms. Define the sequence*

$$c_1 \equiv \Omega tu; \quad c_{n+1} \equiv \Omega(c_n)\Omega.$$

If $\Pi \not\vdash t = u$ then $\Pi \not\vdash c_n = \Omega$ for all n.

Proof. The proof is by induction on n. By Lemma 4.1 we have that $\Pi \vdash \Omega tu = \Omega$ iff $\Pi \vdash t = u$, so that our hypothesis $\Pi \not\vdash t = u$ implies $\Pi \not\vdash c_1 = \Omega$. The remaining part follows from the induction hypothesis and from Lemma 4.1 applied to $c_{n+1} \equiv \Omega(c_n)\Omega$. □

Theorem 4.3 *Every partially ordered model of* Π *is partitioned in an infinite number of connected components (= A-clopen sets), each one containing at most one* λ*-term denotation.*

Proof. Let $\mathbf{C}$ be a partially ordered model of Π. The interpretation of a closed λ-term t is the element $|t|_{\mathbf{C}}$ of $\mathbf{C}$ (see Section 2.2). For the sake of simplicity, we write directly t for $|t|_{\mathbf{C}}$ when there is no danger of confusion. Define $0 \equiv \Omega$ and $s(x,y) \equiv \Omega xy$. Since $\Pi \vdash \Omega xx = \Omega$, then we have that $\mathbf{C} \models \lambda x.\Omega xx = \lambda x.\Omega$. This last identity implies $\Omega cc = (\lambda x.\Omega xx)c = (\lambda x.\Omega)c = \Omega$ for all $c \in C$.

Let t, u be two λ-terms such that $\Pi \not\vdash t = u$. Since $\mathbf{C}$ is a model of Π, by Lemma 4.2 we must have that $\mathbf{C} \not\models c_n = \Omega$ for all $n \geq 1$. Then we can apply Lemma 4.1 to get that t and u are separated by two A-clopen sets. Since we have an infinite number of Π-equivalence classes, then we must have an infinite number of connected components each of them containing at most one term denotation. □

A class $\mathcal{C}$ of models of lambda calculus *represents a lambda theory* $\mathcal{T}$ if there is a model in $\mathcal{C}$ whose theory is exactly $\mathcal{T}$. A class of models is *incomplete* if it does not represent all the lambda theories.

The models of lambda calculus are classified into *semantics* according to the nature of their representable functions. A semantics is usually constituted by a class of suitable partially ordered models. This last condition is justified by Scott's view of models as sets of sets of observations (or informations) and of computable functions as monotone functions over such sets (see [45]).

Scott's continuous semantics [43] is the class of the partially ordered models whose specialization order is a complete partial ordering and the representable functions are all the continuous ones w.r.t. the Scott topology. The graph model semantics (see [44] [18] [34] [35] [9, Section 5.5]) is a subclass of the K-semantics isolated by Krivine (see [28] [9, Section 5.6.2]) within the continuous semantics. The filter model semantics was defined by Coppo, Dezani, Honsell and Longo in [15] (see also [4]) within the continuous semantics.

The stable semantics introduced by Berry [10] is the class of the partially ordered models whose specialization order is a DI-domain and the representable functions are all the stable ones.

The strongly stable semantics introduced by Bucciarelly and Ehrhard in [11] is the class of the partially ordered models whose specialization order is a DI-domain with coherence and the representable functions are all the strongly stable ones. The hypercoherence semantics introduced by Ehrhard [17] is a subclass of the strongly stable semantics.

Stability and strong stability constitute restrictions of continuity to capture the notion of sequentiality.

The first incompleteness result was given by Honsell and Ronchi della Rocca [24] for the continuous semantics. They proved that the contextual lambda theory induced by the set of essentially closed terms does not admit a continuous model. Following a similar method, Gouy [20] proved the incompleteness of the stable semantics. Other more semantic proofs of incompleteness for the continuous and stable semantics can be found in [7]. Bastonero [6] provides an incompleteness result for the hypercoherence semantics. Salibra has recently shown in [37] that the strongly stable semantics is also incomplete.

Theorem 4.4 (The Incompleteness Theorem) *Any semantics of the lambda calculus given in terms of partially ordered models with a finite number of connected components (= minimal upper and lower sets = A-clopen sets) is incomplete. If constants are admitted then, for every cardinal number δ, any semantics of the lambda calculus given in terms of partially ordered models with at most δ connected components is incomplete.*

Proof. From Thm. 4.3. If constants are admitted, it is sufficient to define the lambda theory Π in a language with an arbitrary number of constants. □

It follows from Thm. 4.4 that the lambda theory Π cannot have a model in the graph model semantics, K-semantics, filter model semantics, stable semantics, hypercoherence semantics, strongly stable semantics, and moreover, in the partially ordered models either with a bottom element, or with a top element, or with a structure of complete partial ordering, meet semilattice, join semilattice and lattice.

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